The Shortest Scale of Quantum Field Theory

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Abstract

The vacuum of quantum field theories of a large number of fields can be unstable due to strong vacuum energy fluctuations, decaying rapidly into many small black holes. We propose that in stable theories, the number of independent quantum mechanical degrees of freedom in a given volume is therefore bounded. Our bound can be made consistent with entropy bounds and holography, but does not seem to be equivalent to either. Rather than setting a direct limit on the number of elementary particles it prescribes a definition, in terms of the Planck length and the number of species, of the shortest scale of any quantum field theory compatible with gravity.

The existence of a fundamental value for the entropy of a black hole (BH) [1] which depends on a geometric property, its area, seems startling since it appears to limit the number of different types of elementary particles $\mathcal{N}$. If there were a sufficiently large number of particle species within a given mass scale, then any black hole much larger than that scale could have formed in a sufficiently large number of ways as to exceed any bound on entropy which does not depend on $\mathcal{N}$. This argument goes beyond holography [2,3], which states that the entropy is established at the boundary, but which nevertheless allows that surface-associated entropy to be proportional to $\mathcal{N}$. In fact, according to elegant arguments and calculations by Bombelli *et al* and Srednicki [4], surface entanglement entropy is indeed proportional to $\mathcal{N}$ [5,6]. But why then should $\mathcal{N}$ be limited in any field theory that is to be consistent with gravity? Moreover, if gravity is the limit of a large N gauge theory [7] how could it disallow large $\mathcal{N}$?

We propose that gravitational stability of the vacuum requires that quantum field theories (QFT's) obey a universal bound on the number of degrees of freedom in a given volume. This can be achieved either by limiting the shortest scale (the ultraviolet cutoff) of the theory or by limiting the number of fields, or both. We first show that the vacuum of free QFT's of a large number of fields is potentially unstable. For such theories, vacuum energy fluctuations in regions whose volume is smaller than a certain "critical" volume (which can be parametrically larger than a Planck volume) become so strong that they induce them to collapse and form BH's. The vacuum then rapidly decays into a BH slush. We then show that to prevent this catastrophic decay, theories must have a bounded number of independent quantum mechanical degrees of freedom in a given volume. We note that the prohibition against vacuum decay into real black holes is compatible with previously derived entropy bounds [8] and with holography. The bound we present on the relationship between the shortest scale of a quantum field theory and its number of elementary particle species is expected to be closely related, but not equivalent, to the previous bounds.

Energy fluctuations in the vacuum occur even though the vacuum is an eigenstate of the total hamiltonian. For our concrete discussion of energy fluctuations in the vacuum we consider a free field theory of $\mathcal{N}$ massless bosonic scalar fields, but our results are applicable, with slight modifications, to physical components of any kind of field, such as fermions, gauge bosons, etc.. We assume that the fields' masses are protected from quantum corrections, for example, by supersymmetry, or a gauge symmetry, so they are strictly massless. The restriction to massless fields is mainly for convenience, allowing us to present simpler analytic results which capture the essence of our point. We further assume that the cosmological constant has been set to zero (at least to some accuracy), for example by unbroken supersymmetry. The curvature of spacetime is therefore much lower than Planckian, and we assume for simplicity that the background space is Minkowski space. We do expect, however, that the vacuum instability we will demonstrate persists in presence of a (small) cosmological constant.

Throughout we emphasize functional dependence on mass scales and $\mathcal{N}$ (which we assume to be a large parameter), and work in units in which $\hbar = c = 1$. We will show that, taking into account energy fluctuations in the vacuum, the assumption that background spacetime is flat is inconsistent unless the number of degrees of freedom is bounded.

The hamiltonian $H = \int d^3x\ \mathcal{H}(\vec{x})$, of a single massless scalar field ϕ in Minkowski spacetime is given by a volume integral over the hamiltonian density $\mathcal{H}(\vec{x}) = \frac{1}{2}\left[\left(\vec{\Pi}(\vec{x})\right)^2 + \left(\vec{\nabla}\phi(\vec{x})\right)^2\right]$, where $\vec{\Pi}$ is the momentum conjugate to ϕ. Separating space into two parts, an "inside" region of volume V and an "outside" region of volume $\widehat{V}$, the total hamiltonian is simply given by $H = H_V + H_{\widehat{V}} = \int\limits_V d^3x\ \mathcal{H}(\vec{x}) + \int\limits_{\widehat{V}} d^3x\ \mathcal{H}(\vec{x})$.

Although the vacuum state is an eigenstate of H, it is not an eigenstate of H_V or $H_{\widehat{V}}$. So in spite of the vacuum being an eigenstate of the total Hamiltonian, the energy contained in the volume V is subject to fluctuations, its dispersion given by

$$\left\langle (\Delta H_V)^2 \right\rangle = \frac{1}{8} \int\limits_V d^3y_1\ d^3y_2 \left[\int \frac{d^3p\ d^3q}{(2\pi)^6} \times \left\{ e^{-i(\vec{p}+\vec{q})\cdot(\vec{y}_1 - \vec{y}_2)} \left[pq + 2\vec{p}\cdot\vec{q} + \frac{(\vec{p}\cdot\vec{q})^2}{pq} \right] \right\} \right]. \quad (1)$$

Note that if V in (1) is the whole of space, then the integration over $\vec{y}_1$ and $\vec{y}_2$ produces a $\delta^3(\vec{p}+\vec{q})$, forcing the momentum integral to vanish. The dispersion of $H_{\widehat{V}}$, as expected, is equal to that of H_V. This can verified by expressing $\int_{\widehat{V}}$ as $\int_{IR^3} - \int_V$ for the d^3x and d^3y integrations, and using the fact that each of the $\int_{IR^3}$ integrals gives a vanishing result due to the presence of $\delta^3(\vec{p}+\vec{q})$.

It is convenient to express (1) as an integral of a density, $\left\langle (\Delta H_V)^2 \right\rangle = \int\limits_V d^3y_1 d^3y_2 F(|\vec{y}_1 - \vec{y}_2|)$, where the density of energy fluctuations $F(|\vec{y}_1-\vec{y}_2|)$ is given by the expression inside the square brackets on the r.h.s. of (1). Since F depends only on $x \equiv |\vec{y}_1-\vec{y}_2|$, we can perform all the integrals in (1), except for the x integral, by using the equality $1 = \int_0^\infty dx\delta(|\vec{y}_1-\vec{y}_2|-x)$. The result is the following convolution,

$$\left\langle (\Delta H_V)^2 \right\rangle = \int dx F(x) \mathcal{D}_V(x) \,, \tag{2}$$

where the geometric factor $\mathcal{D}_V(x)$ depends only on the shape of the volume V.

The energy dispersion calculation leading to (1),(2), has many similarities to the calculation of the entanglement entropy of a subsystem of a pure state [4]. Since the dispersion of H_V is equal to that of $H_{\widehat{V}}$ they can depend only on properties of the common boundary of the two regions. This is perhaps counterintuitive, one might have expected the dispersion of H_V to be extensive, proportional to the volume V, but, as the previous argument shows, this is wrong. The fact that the dispersion of H_V has to be a function of boundary invariants and using dimensional analysis allows us to estimate it in different setups. Of course, as it stands, $\left\langle (\Delta H_V)^2 \right\rangle$ is ultraviolet divergent, being an operator of mass dimension 2; to define it we have to introduce an ultraviolet momentum cutoff Λ. The exact form of implementing the cutoff will not affect the nature of our results, but it will change details, such as numerical coefficients.

For the sake of concreteness and clarity, we restrict our attention for the moment to the case of a spherical volume V of radius R. We expect similar results when different geometries are considered, and present some examples later on. On dimensional grounds, the energy dispersion in a sphere of radius R, $\Delta E(R) = \sqrt{(\Delta H_{Sphere})^2}$, is given by $\Delta E(R) = \mathcal{E}(R\Lambda)\Lambda$. We now proceed to find the analytical expression for the function $\mathcal{E}(R\Lambda)$. The geometric factor for a spherical volume is given by

$$\mathcal{D}_{Sphere}(x,R) = \frac{\pi^2}{3} x^2 (x-2R)^2 (x+4R) \quad 0 < x < 2R\,, \tag{3}$$

and, of course, vanishes for $x > 2R$. Since the density F is ultraviolet divergent, it has to be regularized. We implement a particularly simple regularization procedure (other regularization choices will be discussed later) by inserting factors of $e^{-p/\Lambda}$ and $e^{-q/\Lambda}$ which suppress momenta larger than Λ in the momentum integrals of eq. (1). Now we can explicitly evaluate F,

$$F(x,\Lambda) = \frac{\Lambda^8}{2\pi^4} \frac{3 - 10(\Lambda x)^2 + 3(\Lambda x)^4}{\left(1+(\Lambda x)^2\right)^6}. \tag{4}$$

Notice that F has an over all factor of Λ^8 as required by its dimensionality, that the maximal value of F is at zero $F(0,1) = \frac{3}{2\pi^4} \sim 0.015$, and that for large x, F is positive and decreases as x^{-8}. Using (3) and (4), integral (2) for ΔE can be evaluated explicitly,

$$\Delta E(\Lambda, R) = \frac{(\Lambda R)^3}{\pi} \left[\frac{8\left[(5+4(\Lambda R)^2\right]}{15\left[1+4(\Lambda R)^2\right]^3} \right]^{1/2} \Lambda\,. \tag{5}$$

For regions of different shapes and different cutoff procedures we expect similar results and indeed have found similar results, as can be seen from Fig. 1.

In a theory of a large number of fields $\mathcal{N}$, the energy dispersion $(\Delta E_{\mathcal{N}}(R))^2 = \left\langle (\Delta H_{Sphere})^2 \right\rangle$ is proportional to $\mathcal{N}$, since the contribution of each field adds up linearly, so $\Delta E_{\mathcal{N}}(R)$ is given by

$$\Delta E_{\mathcal{N}}(R) = \sqrt{\mathcal{N}}\, \mathcal{E}(R\Lambda)\Lambda\,, \tag{6}$$

where $\mathcal{E}(R\Lambda)$ can be read off (5). Note that $\Delta E_{\mathcal{N}}(R)$ has some very different properties than the expectation value of vacuum energy $\langle H \rangle$ (the cosmological constant). For example, bosonic and fermionic fields contribute with different signs to $\langle H \rangle$, so an exact cancelation, as in a supersymmetric theory, is possible. But all the contributions to the dispersion have the same sign, and cancellation is not possible. In addition, their $\mathcal{N}$ dependence is different, $\Delta E_{\mathcal{N}}(R)$ being proportional to $\sqrt{\mathcal{N}}$, while $\langle H \rangle$ is generically proportional to $\mathcal{N}$. Moreover, since we are dealing with fluctuations, it is clear that $\Delta E_{\mathcal{N}}(R)$ should not be considered as ordinary, classical energy, but rather as a stochastic quantity.

We would like to show that unless the number of degrees of freedom of QFT's is bounded their vacuum is gravitationally unstable. Let us consider a theory of $\mathcal{N}$ massless scalar fields in a classical spacetime background. If spacetime curvature is smaller than Planckian, then the energy-momentum tensor of the QFT can be consistently used as a source in the classical Einstein equations for the metric. Since the expectation value of the energy-momentum tensor vanishes (recall that we have assumed that the cosmological constant vanishes), we consider its fluctuations as a stochastic source in the Einstein equations [9]. Taking $\Delta E_{\mathcal{N}}(R)$ of eq. (5) as a source in Einstein's equations, we immediately encounter a potential problem. When a typical energy fluctuation is within its own Schwarzschild radius

$$G_N \Delta E_{\mathcal{N}}(R) \gtrsim R \tag{7}$$

a BH could be created (G_N is Newton's constant). If indeed BH's are created, they are created at a rate of about Λ and at a density of about close packing, making the vacuum of the theory very unstable.

That a fluctuation is within its own Schwarzschild radius is not sufficient information to determine whether a BH would actually be formed. An additional necessary condition is that the size of the created energy lump is such that all its different parts are in causal contact, that is, the travel time of light through the collapsing region must be comparable to the mean lifetime of the energy fluctuation itself, which can be estimated to be $t \sim (\Delta E_1)^{-1}$. This results in the condition $R\Delta E_1(R,\Lambda) \lesssim 1$ which, as we will see later, is satisfied in our examples if $R\Lambda \lesssim 5$. The volume of the forming BH's can therefore be a few hundreds "unit cells" (of volume Λ^3), which make their treatment as field theoretic object a valid approximation. As we will also see later, for a large enough $\mathcal{N}$, the size of created BH's is large in Planck units as well, so the initial induced curvature by each one of them is small. We conclude that in this case backreaction is expected to be quite small initially, so that using a flat spacetime background to estimate rate of formation of BH's, as we do, is a valid approximation. In addition, to be able to treat the forming BH's as classical objects, as we do, their evaporation time by emitting Hawking radiation $t_{\rm ev} \simeq 10^4 \frac{M^3 G_N^2}{\mathcal{N}}$, should be longer than their characteristic classical time scale $t_{\rm cl} = G_N M$. By setting $M = \Delta E_{\mathcal{N}}$, we find that $t_{\rm ev} > t_{\rm cl}$ if $G_N \Lambda^2 \gtrsim 10^{-4} \frac{1}{\mathcal{E}(\Lambda R)^2}$. Note that this condition does not depend on $\mathcal{N}$. As we will show later, this limits the range of validity of our discussion to theories for which $G_N \Lambda^2 \gtrsim 1/500$.

The instability of flat space to the formation of a BH slush that we have just discovered can be avoided if BH formation is guaranteed not to occur, that is, condition (7) should not be satisfied for any value of R. Substituting (6) into (7) we find that this is so if

$$G_N \sqrt{\mathcal{N}} \Lambda^2 \lesssim \frac{R\Lambda}{\mathcal{E}(R\Lambda)} \tag{8}$$

is satisfied for all values of R[1]. In practice, to be sure that BH's are not created, bound (8) needs to be somewhat stronger, such that the probability of forming a BH is much smaller than unity. As it turns out, it is enough to check condition (8) for R's such that $R\Lambda\mathcal{E}(\Lambda R) \sim 1$. The numerical factor $\alpha^2 \equiv \left.\frac{R\Lambda}{\mathcal{E}(R\Lambda)}\right|_{R\Lambda\mathcal{E}(\Lambda R)=1}$, appearing in condition (8) can be estimated from eq. (5) for the specific case of a spherical volume and exponential momentum cutoff (we have found in this case $\alpha^2 \sim 17$). We have analyzed other regularization schemes and the results are similar. The results for exponential, gaussian, and sharp cutoffs are shown in Fig. 1. We have also calculated explicitly the geometric factor $\mathcal{D}$ for cubic volumes, and present the final result for a cube with the same volume as sphere of radius R in Fig. 1. Condition (8) may not be the tightest, since when $\mathcal{N}$ is not large enough, our estimates become inaccurate, but it is enough to make our point.

Finally it is straightforward to rewrite condition (8) as a bound on the ultraviolet cutoff of the theory

$$\Lambda \lesssim \frac{\alpha}{\mathcal{N}^{1/4}} M_p \,, \tag{9}$$

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[1]To be more precise, since condition (8) was derived using some approximations, it has to be satisfied for all values of R, Λ and $\mathcal{N}$, such that the approximations used in deriving (8) are valid.

where we have introduced the Planck mass $M_p = 1/\sqrt{G_N}$.

We have found that for a given Λ, treating $\mathcal{N}$ as a variable, there is a critical value above which the vacuum becomes gravitationally unstable. Alternatively, if $\mathcal{N}$ is fixed and we treat Λ as a variable, we find that Λ cannot be made larger than a certain critical value, which is parametrically lower than the Planck scale. Conversely, a large number of massless fields $\mathcal{N} \sim 10^4$ (about the number of massless modes in some string theories) can be admitted in a field theory provided that the ultraviolet cutoff of the theory (e.g. the string scale) is somewhat below the Planck scale.

In effect, as Λ or $\mathcal{N}$ are increased, the number of degrees of freedom in a given volume increases, and we have found that whenever this number increases beyond a certain critical value, a gravitational instability turns on. We have concluded that in order to avoid a disastrous "granular collapse" of spacetime, the number of fundamental degrees of freedom has to be bounded. Note the instability occurs without reference to quantum gravitational effects. Because our calculation assumes a flat background, we have not explicitly derived the mechanism by which the scale of QFT is cut off, nor even proved that such a mechanism could be understood in the context of QFT. We believe that the physics behind our argument is not unconnected to the constraints on the number of particles species that come from string theory or that appear to be implied by entropy considerations and holography (see below). On the other hand, we have obtained our result without any reference to strings or entropy, and that raises the intriguing possibility that such implications of string theory may be more general than the theory itself.

The QFT cutoff Λ determines a single "bit" of information area $A_{SIB} \sim 1/\Lambda^2$. We may now ask whether the size of A_{SIB} given by condition (9) is compatible with the proposed statistical explanation of BH entropy [1] as given entirely by entanglement entropy [4]. Recall that the entropy of a BH is proportional to its horizon area A_H in units of Newton's constant, $S_{BH} = A_H/4G_N$ and does not depend on $\mathcal{N}$, while entanglement entropy $S_{EN} = \mathcal{N} A/A_{SIB}$ depends linearly on $\mathcal{N}$. Considering S_{BH} and S_{EN} together in a way as to make them

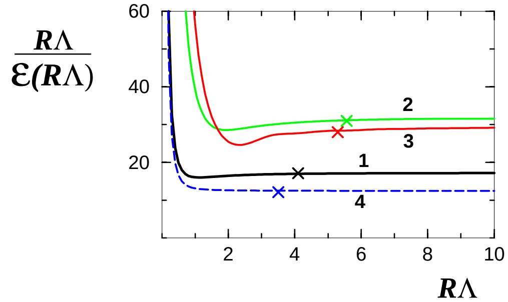

FIG. 1. The function $\frac{R\Lambda}{\mathcal{E}(R\Lambda)}$ for a spherical volume of radius R, and different regularizations (labeled 1,2,3), and for a cube of the same volume (labeled 4). Crosses indicate points for which $R\Lambda\mathcal{E}(R\Lambda)$=1.

compatible without any bound on $\mathcal{N}$, would suggest that A_{SIB} should be proportional to $\mathcal{N}$. However, condition (9) suggests that the cutoff area $1/\Lambda^2$ scales only as $\mathcal{N}^{1/2}$. Thus, if the QFT cutoff determines the true size of a single information bit, we are left with an upper bound on $\mathcal{N}$. This upper bound is not as strong as what one would obtain [10] by consideration of the entropy of a BH at the "naive" cutoff, namely the Planck scale, which admits $\mathcal{N}$ only somewhat greater than unity. Our calculation suggests that we can still reconcile the idea that BH entropy is all due to entanglement entropy with the existence of a large number of species, provided that the area of a single bit of information is larger than the Planck area L_p^2.

We would like to compare our analysis to vacuum entropy considerations [4]. Obviously the entropy of the vacuum vanishes, since it is a pure state. There are different possibilities to define entropy in a given volume V. First, we may define it as entanglement entropy. This is equivalent to putting boundary conditions on the boundary of the volume V. Then, as shown by Bekenstein [6] for a single field, if an upper bound on E is imposed, and entanglement entropy is calculated given that information, the Bekenstein entropy bound on entropy $S \lesssim ER$, is obeyed. For $\mathcal{N}$ fields, the situation is very similar, except that both S and E are proportional to $\mathcal{N}$, so the ratio S/E does not depend on $\mathcal{N}$. But we have chosen to put boundary conditions at infinity, and leave the values on the boundary of V completely free. We could compute the entropy of a typical perturbation, given that it was formed. This amounts to taking the logarithm of the number of ways in which a typical energy fluctuation could have been formed, which is simply (when $R \gtrsim 1/\Lambda$) $S \sim R^3 \int d^3k \sim 4\pi(\Lambda R)^3$. To find whether Bekenstein's entropy bound is obeyed we need to compare S and $R\Delta E = R\Lambda\mathcal{E}(R\Lambda)$. We observe that $S \lesssim \Delta E R$ requires $\Lambda R \lesssim 1$. So typical energy fluctuations *do not* obey the Bekenstein bound or for that matter the holography bound. This is fine, since these are unstable configurations, which live only for a period of time $t \sim \Delta E_1^{-1}$. This demonstrates the importance of the principle that constitutes the basis of our argument: that actual black holes must not form out of virtual vacuum fluctuations.

Finally, an appealing consistency between our bound and string theory is that indeed in string theories that allow a large number of massless fields in their perturbative low energy spectrum, the string scale α' which provides the effective field theory cutoff is somewhat below the Planck scale as required by (9).

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